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ON THE MOTION OF THE HESS GYROSCOPE

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Iu. A. ARKHANGEL'SKII
(Moscow)

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As we know, the solution of the equations of motion of a heavy rigid body about a fixed point in the Hess' case

$$e_1 \sqrt{A(B-C)} + e_3 \sqrt{C(A-B)} = 0, \quad e_2 = 0$$

(A, B, C are the principal moments of inertia and e_1, e_2, e_3 are the coordinates of the center of gravity of the body) is not reducible to quadratures; it is reduced to the solution of the Ricatti differential equation. This complicates investigation of the corresponding motion considerably.

A general qualitative pattern of motion of a body was first given for the Hess' case by N. E. Zhukovskii [1] and followed in more detail by Kovalev [2, 3] who employed the method of moving hodograph (*).

However both these geometrical interpretations are fairly complicated, and give rise to severe difficulties when it comes to determining the motion of a specific rigid body under concrete initial conditions.

In the present paper we study the Hess' case of the motion of a rigid body under the assumption that at the initial instant a high angular velocity ω_0 about some axis, is imparted to the body. We obtain explicit relations connecting the Euler angles with time, and these enable us to analyze in detail the motion of the Hess gyroscope without much difficulty.

1. We construct the equations of motion of a rigid body in the associated rectangular coordinate system $Oxyz$ whose Ox-axis passes through the center of gravity of the body, while the Oy and Oz axes are chosen in such a manner (this is always possible in the Hess case [4]) that the expression for the kinetic energy of the body becomes

$$2T = a_1 x^2 + a(y^2 + z^2) - 2byz$$
$$a_1 = A_{11}(A_{22}A_{33})^{-1}, \quad a = A_{33}^{-1}, \; b = A_{13}(A_{11}A_{33})^{-1}$$

Here x, y, z are the projections of the kinetic moment of the body on the $Oxyz$ axes, and A_{11}, A_{22}, A_{33}, A_{13} are the components of the corresponding inertia tensor for which the relation $A_{13}^2 = A_{11}(A_{22} - A_{33})$ holds.

We also note the following expressions for the projections ω_1, ω_2, ω_3 of the angular velocity:

$$\omega_1 = -by, \quad \omega_2 = ay, \quad \omega_3 = az$$

*) See also A. M. Kovalev's "Geometrical investigation of certain solutions of the problem of motion of a body with a fixed point", Candidate's dissertation, Donetsk State Univ., 1969.

In the Hess' case the fourth integral is $x = 0$ in the $Oxyz$ system while the equations themselves, their first three integrals, and the projections of the angular velocity under the condition $b > 0$ in the dimensionless variables

$$y = \sqrt{\Gamma / b}\, y_1, \quad z = \sqrt{\Gamma / b}\, z_1, \quad t = t_1 / \sqrt{\Gamma b}, \quad \omega_i = a \sqrt{\Gamma / b}\, \omega_i^*$$
$$\Gamma = \mathrm{Mg} \sqrt{e_1^2 + e_3^2}, \qquad i = 1, 2, 3 \tag{1.1}$$

assume the form [2]

$$\frac{dy_1}{dt_1} = -y_1 z_1 - \gamma_3, \qquad \frac{dz_1}{dt_1} = y_1^2 + \gamma_2 \tag{1.2}$$

$$\frac{d\gamma_1}{dt_1} = \frac{2}{c}(z_1\gamma_2 - y_1\gamma_3), \quad \frac{d\gamma_2}{dt_1} = -y_1\gamma_3 - \frac{2}{c} z_1\gamma_1, \quad \frac{d\gamma_3}{dt_1} = y_1\left(\gamma_2 + \frac{2}{c}\gamma_1\right) \tag{1.3}$$

$$y_1^2 + z_1^2 - c\gamma_1 = ch, \; y_1\gamma_2 + z_1\gamma_3 = k_1, \; \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.4}$$

$$\omega_1^* = (-c / 2)\, y_1, \quad \omega_2^* = y_1, \quad \omega_3^* = z_1 \tag{1.5}$$

Here $c = 2b / a$; $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the vertical, and h and k_1 are dimensionless arbitrary constants.

Using the polar coordinates ρ and α

$$y_1 = \rho \cos \alpha, \quad z_1 = \rho \sin \alpha \tag{1.6}$$

to rewrite the energy and surface integrals in the form

$$\rho^2 = c(\gamma_1 + h), \qquad \rho(\gamma_2 \cos \alpha + \gamma_3 \sin \alpha) = k_1 \tag{1.7}$$

we obtain (applying (1.2) and (1.3)) the following equations for γ_1 and w:

$$w = \operatorname{tg} {}^1/_2\, \alpha \tag{1.8}$$

$$c\frac{d\gamma_1}{dt_1} = 2 \operatorname{sign}(\gamma_{20} \sin \alpha_0 - \gamma_{30} \cos \alpha_0) \sqrt{c(h + \gamma_1)(1 - \gamma_1^2) - k_1^2}$$

$$2\frac{dw}{dt_1} = (1 - w^2)\sqrt{c(h + \gamma_1)} + k_1 [c(h + \gamma)]^{-1}(1 + w^2)$$

Here and below we shall denote any function $F(t)$ by $F_0 = F(0)$.

We assume that at the initial instant the axis Ox lies in the horizontal plane, the Oz axis is inclined to the vertical at an angle θ_0 $(0 < \theta_0 < {}^1/_2\pi)$ and that a high angular velocity ω_0 about this axis is imparted to the body. Then the initial conditions are

$$\gamma_{10} = 0, \quad \gamma_{20} = \sin \theta_0, \; \gamma_{30} = \cos \theta_0$$
$$y_{10} = 0, \quad z_{10} = \omega_0^*, \quad \rho_0 = \omega_0^*, \; \alpha = \pi/2 \tag{1.9}$$

and the relations (1.7) yield

$$\omega_0^{*2} = \mathrm{ch}, \; k_1 = \omega_0^* \cos \theta_0 \tag{1.10}$$

Passing to the new variables σ and τ and introducing a small parameter μ

$$\gamma = h\sigma, \quad \tau = t_1 \sqrt{h / c}, \quad \mu = h^{-1} \tag{1.11}$$

we obtain

$$\frac{d\sigma}{d\tau} = 2\sqrt{f(\sigma)}, \quad f(\sigma) = (1 + \sigma)(\mu^2 - \sigma^2) - \mu^2 \cos^2 \theta_0 \; (\sigma_0 = 0) \tag{1.12}$$

$$2\frac{dw}{d\tau} = c(1 - w^2)\sqrt{1 + \sigma} + \mu(1 + w^2)\cos \theta_0 (1 + \sigma)^{-1} \tag{1.13}$$

2. To solve the equation (1.12) we find the roots [5] $\sigma_1, \sigma_2, \sigma_3$ of the cubic equation $f(\sigma) = 0$ which are

$$\sigma_1 = \mu \sin \theta_0 + {}^1/_2\mu^2 \cos^2\theta_0 + \mu^3(\ldots), \; \sigma_2 = -\mu \sin \theta_0 + {}^1/_2\mu^2 \cos^2 \theta_0 + \mu^3(\ldots) \tag{2.1}$$
$$\sigma_3 = -1 - \mu^2 \cos^2 \theta_0 + \mu^3(..)$$

and performing the substitution

$$\sigma = \sigma_1 - (\sigma_1 - \sigma_2)\,\xi^2$$

$$\xi_0 = \sqrt{\frac{\sigma_1}{\sigma_1 - \sigma_2}} = \frac{1}{\sqrt{2}}\left[1 + \frac{\mu}{4}\frac{\cos^2\theta_0}{\sin\theta_0} + \mu^2(\ldots)\right] \tag{2.2}$$

we obtain

$$\frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} = -\sqrt{\sigma_1 - \sigma_3}\,d\tau \quad \left[k^2 = \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} = 2\mu\sin\theta_0 + \mu^2(\ldots)\right] \tag{2.3}$$

Inverting the corresponding elliptic integral we obtain

$$\xi = \mathrm{sn}\,(\tau^{(0)} - \sqrt{\sigma_1 - \sigma_3}\,\tau) \quad (\xi_0 = \mathrm{sn}\,\tau^{(0)})$$

as a function periodic in τ of the period T_1

$$T_1 = 4K\,(\sigma_1 - \sigma_3)^{-1/2} = 2\pi T_2\,[T_2 = 1 - {}^1\!/_4\mu^2\sin^2\theta_0 + \mu^3(\ldots)] \tag{2.4}$$

where K is a complete elliptic integral of the first kind.

Introducing a new variable τ^* in place of τ

$$\tau = T_2\tau^* \tag{2.5}$$

we find that the quantity

$$\xi = \mathrm{sn}\,(\tau^{(0)} - 2K\,\pi^{-1}\tau^*) \tag{2.6}$$

is a 2π-periodic function of τ^*.

Expanding ξ by means of the formula

$$\mathrm{sn}\,u = \sin u_1\,[1 + {}^1\!/_8\,k^2\,(1 + \cos 2u_1)] + k^4(\ldots), \quad u_1 = {}^1\!/_2\,\pi u K^{-1}$$

together with the expressions for k^2 and ξ_0 quoted above and inserting the result into the power series in μ given in (2.15), we obtain

$$\sigma = \mu\sin\theta_0\sin 2\tau^* + {}^1\!/_2\mu^2\,(\cos^2\theta_0 - {}^1\!/_2\sin^2\theta_0 - \cos 2\theta_0\cos 2\tau^* - {}^1\!/_2\sin^2\theta_0\cos 4\tau^*) + \mu^3(\ldots) \tag{2.7}$$

We note that $\sigma(\tau^*)$ is a π-periodic function of τ^*.

Using the relations (2.5) and (2.7) we can rewrite (1.13) to obtain

$$2\frac{dw}{d\tau^*} = c\,(1 - w^2) + \mu\,[{}^1\!/_2\,c\,(1 - w^2)\sin\theta_0\sin 2\tau^* + (1 + w^2)\cos\theta_0] + \mu^2\,[(1 - w^2)(\ldots) - {}^1\!/_2\,(1 + w^2)\sin 2\theta_0\sin 2\tau^*] + \mu^3(\ldots) \tag{2.8}$$

the right-hand side of which is a π-periodic function of τ^*. This equation has a unique π-periodic solution W which can be written in the form of a series

$$W = w^* + \mu w_1 + \mu w_2 + \ldots$$

with periodic coefficients given by

$$2\frac{dw^*}{d\tau^*} = c\,(1 - w^{*2}) \tag{2.9}$$

$$\frac{dw_j}{d\tau^*} = -cw_j + F_j\,(\tau^*, w^*, w_1, \ldots, w_{j-1}) \quad (j = 1, 2, \ldots) \tag{2.10}$$

where F_j is a π-periodic function of τ^*.

In fact, $w^* = 1$ is the only periodic solution of (2.9) and each of the Eqs. (2.10) has also a unique periodic solution [6] by virtue of the fact that c is real.

Thus, we obtain the following expression for W

$$W = 1 + \mu\frac{\cos\theta_0}{c} + \mu^2\left[\frac{\cos^2\theta_0}{2c^2} - \frac{3\sin 2\theta_0}{4(c^2+4)}(c\sin 2\tau^* - 2\cos 2\tau^*)\right] + \mu^3(\ldots) \tag{2.11}$$

and writing for it a variational equation, we confirm that for sufficiently small μ the periodic solution (2.11) is asymptotically stable. As for the required particular solution ($w_0 = 1$) of (2.8) (which, generally speaking, can be found using the Poincaré theorem [6]), we find that by virtue of the fact that

$$|w - W| \sim \mu \exp(-c\tau^*), \qquad \tau^* \sim \mu^{-1/2} t$$

it converges very rapidly to W. For this reason we shall use the solution (2.11) instead of the solution of (2.8).

Formulas (1.8) and (2.11) yield

$$\sin\alpha = 1 - {}^1/_2\, \mu^2 c^{-2} \cos^2\theta_0$$
$$\cos\alpha = -\mu c^{-1}\cos\theta_0 + {}^3/_4 \mu^2 (c^2+4)^{-1} \sin 2\theta_0 (c \sin 2\tau^* - 2\cos 2\tau^*)$$

which, on insertion into (1.6) and using (1.7), (1.11) and (2.7), give

$$y_1 = \left(\frac{c}{h}\right)^{1/2} \left\{-\frac{\cos\theta_0}{c} + \mu\left[\frac{3\sin 2\theta_0}{4(c^2+4)}(c\sin 2\tau^* - 2\cos 2\tau^*) - \frac{\cos\theta_0}{c}\right] + \mu^2(\ldots)\right\}$$
$$z_1 = \sqrt{ch}\,\{1 + {}^1/_2 \mu \sin\theta_0 \sin 2\tau^* + {}^1/_8 \mu^2 [2\cos 2\theta_0 (1 - \cos 2\tau^*) + \sin^2\theta_0 \sin^2 2\tau^*] + \mu^3(\ldots)\} \qquad (2.12)$$

Formulas (1.2), (1.11), (2.20) and (2.25) yield

$$\gamma_1 = \sin\theta_0 \sin 2\tau^* + {}^1/_2\, \mu\, [\cos^2\theta_0 - {}^1/_2 \sin^2\theta_0 - \cos 2\theta_0 \cos 2\tau^* - {}^1/_2 \sin^2\theta_0 \cos 4\tau^*] + \mu^2(\ldots)$$
$$\gamma_2 = \sin\theta_0 \cos 2\tau^* - \mu(c^{-1}\cos^2\theta_0 - {}^1/_2 \cos 2\theta_0 \sin 2\tau^*) + \mu^2(\ldots) \qquad (2.13)$$
$$\gamma_3 = \cos\theta_0 + \mu\cos\theta_0 (1 - \sin\theta_0 \sin 2\tau^*) + \mu^2(..)$$

3. We shall analyze the motions of the Hess' gyroscope using the Euler angles θ, φ, ψ

$$\cos\theta = \gamma_3, \quad \frac{d\psi}{dt} = \frac{\omega_1\gamma_1 + \omega_2\gamma_2}{1-\gamma_3^2}, \qquad \frac{d\varphi}{dt} = \omega_3 - \frac{d\psi}{dt}\cos\theta \qquad (3.1)$$

The first formula of (2.14), the last one of (2.13) as well as (1.1) and (1.11) yield

$$\theta = \theta_0 - 2c\omega_0^{-2}\lambda_1 (1 - \sin\theta_0 \sin\omega_0 t) + \omega_0^{-4}(\ldots) \quad (\lambda_1 = ac^{-1}\Gamma \operatorname{ctg}\theta_0) \qquad (3.2)$$

Inserting (1.1), (1.5), (2.12) and (2.13) into the second relation of (3.1) and integrating, we obtain the following expression for the angle of precession

$$\psi = \psi_0 - \omega_0^{-2}\lambda_1 (2\sin\omega_0 t + c\cos\omega_0 t) + \omega_0^{-3}\lambda_2 t + \omega_0^{-4}(\ldots) \qquad (3.3)$$
$$\lambda_2 = \lambda_1^2 \sec\theta_0 [(c^2+4)\cos^2\theta_0 - 2c^2]$$

Formulas (1.1), (1.5), (1.11), (2.12), (2.13), (3.1) and (3.3) yield an expression for the angle of self-rotation in the form

$$\varphi = \omega_0 t + \omega_0^{-2}\lambda_1 \sec\theta_0 [c\cos 2\theta_0 (\cos\omega_0 t - 1) + 2\cos^2\theta_0 \sin\omega_0 t] + \omega_0^{-3}(\ldots) \qquad (3.4)$$

We determine the motion of the Hess' gyroscope using the formulas (3.2)–(3.4) as follows. On a unit sphere with center at the fixed point, we construct a spherical rectangle by taking two parallels separated from the mean parallel $\theta_0 - 2\omega_0^{-2}c\lambda_1$ by the angle of $\pm 2\omega_0^{-2} c\lambda_1 \sin\theta_0$, and two meridians separated from the mean meridian ψ_0 by the angle of $\pm\omega_0^{-2}\lambda_1(4+c^2)^{1/2}$. Let now the unit sphere rotate about the vertical with a constant low angular velocity $\omega_0^{-3}\lambda_2$. Then the Oz axis traces a trajectory ($\theta_1 = \theta - \theta_0 + 2\omega_0^{-2}c\lambda_1$, $\psi_1 = \psi - \psi_0$) on the unit sphere, and this trajectory is an ellipse given by

$$\frac{(\theta_1 + c\sin\theta_0\psi_1)^2}{(\lambda_3 c)^2} + \frac{\theta_1^2}{(2\lambda_3)^2} = 1 \qquad (\lambda_3 = \omega_0^{-2} c\lambda_1 \sin\theta_0)$$

inscribed in the spherical rectangle defined above, its principal axes inclined to the θ_1

and ψ_1 axes at the angle β

$$\operatorname{tg} 2\beta = 8c \sin \theta_0 (c^2 + 4 - 4c^2 \sin^2 \theta_0)^{-1}$$

While describing this ellipse the Oz axis of the gyroscope executes in the first approximation a periodic motion of period $T^* = 2\pi\omega_0^{-1}$ touching at the instants $t^{(1)}$ and $t^{(2)}$

$$t^{(1)} = \omega_0^{-1}(\pi n + \varepsilon), \quad t^{(2)} = (2\omega_0)^{-1}(2n+1)\pi \quad (\operatorname{tg} \varepsilon = 2/c, \quad n = 0, \pm 1, \pm 2, \ldots)$$

the side meridians and parallels and intersecting at the instants $t^{(3)}$ and $t^{(4)}$

$$t^{(3)} = {}^1/_2\, \omega_0^{-1}[(2n+1)\pi + 2\varepsilon], \qquad t^{(4)} = \omega_0^{-1}\pi n$$

the mean meridian $(\theta_1 = 0)$ and the mean parallel $(\psi_1 = 0)$.

It follows from formula (3.4) that the self-rotation of the body differs little from the uniform rotation taking place at a high angular velocity ω_0.

The above analysis makes possible a sufficiently detailed investigation of the motion of the Hess' gyroscope and of the dependence of this motion on its initial value and on the constructional parameters of the gyroscope.

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Editorial Note. The author of this paper quotes the following numbers of formulas: (2.14), (2.15), (2.20) and (2.25) which do not appear in the text. They are obviously erroneous quotations which will perhaps be corrected by the author in one of the following issues of the Journal.